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Parabolic equations of Von Karman type on Kähler manifolds

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Abstract

We study a parabolic version of a system of Von Karman type on a compact Kähler manifold of arbitrary dimension. We provide local in time regular solutions, which can be extended to global bounded ones if the data of the problem are small.

Résumé

On étudie un problème parabolique associé à un système elliptique de type Von Karman sur une variété Kählérienne compacte. On prouve l'existence de solutions locales en temps admettant divers types de régularité. On montre qu'on peut les prolonger en solutions globales bornées si les données sont suffisamment petites.

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1. Introduction

1.1. Summary

We consider an evolution problem of parabolic type, related to an elliptic system of Von Karman's equations on a compact Kähler manifold. In [4], we studied the stationary case, using a variational method suggested by Berger's paper [3]. Here, we prove the unique local solvability of the initial value problem, corresponding to arbitrary data, in a class of sufficiently regular functions. If the data are small enough, we prove that such local solutions can be extended to global, bounded ones. Local solutions are obtained by a nontrivial application of a linearization and fixed point technique, as e.g. in Kato [5]; global ones are obtained by a standard continuation method, by means of a time-independent a priori estimate, which can be established on local solutions if the data are sufficiently small. In a future paper, we will consider the existence of global, weak solutions, as well as less regular ones, for which the limit case of the Sobolev imbedding plays a more drastic role than in the present work.

1.2. The equations

All functions we consider are real valued. Let $V = (V_{2m}, g)$ denote a C^∞ compact Kähler manifold, of complex dimension $m \geqslant 2$, without boundary. If $u_1, \dots, u_m \in C^\infty(V)$, we define

$$N(u_1, \dots, u_m) := \delta^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m} \nabla^{\beta_1}_{\alpha_1} u_1 \cdots \nabla^{\beta_m}_{\alpha_m} u_m, \tag{1.1}$$

$$M(u) := N(u, \dots, u) = m! \det(\nabla^\beta_\alpha u), \tag{1.2}$$

where $\delta^{\alpha_1 \dots \alpha_m}_{\beta_1 \dots \beta_m}$ is the Kronecker tensor, and $\nabla^\beta_\alpha := g^{\beta\bar{\gamma}} \partial_{\alpha\bar{\gamma}}$, in any local chart compatible with the complex structure of V. We also set $\Delta u := -\nabla^\alpha_\alpha u$, and adopt the convention

$$N(u_1^{(k_1)}, \dots, u_p^{(k_p)}) := N(\underbrace{u_1, \dots, u_1}_{k_1 \text{ factors}}, \dots, \underbrace{u_p, \dots, u_p}_{k_p \text{ factors}}), \tag{1.3}$$

with $k_1 + \dots + k_p = m$.

Given $T \in]0, +\infty[$, a source term φ defined on $[0, T] \times V$ (or on $[0, +\infty[\times V$), and an initial value u_0 defined on V, we seek to determine on $[0, T] \times V$ (respectively, on $[0, +\infty[\times V$), two functions u and f, satisfying the system

$$u_t + \Delta^m u = N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u), \tag{1.4}$$

$$\Delta^m f = -M(u), \tag{1.5}$$

with u subject to the initial condition

$$u(0, \cdot) = u_0. \tag{1.6}$$

We refer to problem (1.4)+(1.5)+(1.6) as "problem (P)".

1.3. Function spaces

For $1 \leqslant p \leqslant +\infty$, we set $L^p := L^p(V)$, and denote its norm by $|\cdot|_p$. For any integer $k \geqslant 0$, we denote by $W^{k,2}(V)$ the Sobolev space of the measurable functions on V, whose generalized derivatives of order up to k are in L^2, and set

$$H^k := \left\{ u \in W^{k,2}(V) \;\Big|\; \int_V u \, \mathrm{d}x = 0 \right\}. \tag{1.7}$$

The zero-average condition allows us to choose in H^k the norm

$$\|u\|_k := \begin{cases} |\Delta^{k/2} u|_2 & \text{if } k \text{ even}, \\ |\nabla \Delta^{(k-1)/2} u|_2 & \text{if } k \text{ odd}, \end{cases} \qquad u \in H^k, \tag{1.8}$$

and we denote by $\langle \cdot, \cdot \rangle$ the corresponding scalar product in H^0.

We shall need the following two properties of the spaces H^k, $k \geqslant 0$ (see e.g. Adams [1], or Aubin [2]).

(S1) The imbedding:

$$H^{m-h} \hookrightarrow L^{2m/h}, \quad 1 \leqslant h \leqslant m; \tag{1.9}$$

in particular, $H^{m-1} \hookrightarrow L^{2m}$ and $H^{m-2} \hookrightarrow L^m$. (1.9) means that the inequality

$$|u|_{2m/h} \leqslant C_S \|u\|_{m-h} \tag{1.10}$$

holds for all $u \in H^{m-h}$, with C_S independent of u.

(S2) Interpolation inequalities. For $0 \leqslant k_1 \leqslant k \leqslant k_2$, and $u \in H^{k_2}$,

$$\|u\|_k \leqslant C_I \|u\|_{k_1}^{1-\theta} \|u\|_{k_2}^{\theta}, \quad \theta := \frac{k - k_1}{k_2 - k_1}, \tag{1.11}$$

with C_I independent of u.

In the sequel, when we want to be precise, C_H, C_S or C_I will denote constants related to Hölder, Sobolev or interpolation inequalities.

Given $T > 0$ and a Banach space X, we denote by $C([0,T]; X)$ the space of the continuous functions from $[0,T]$ into X, endowed with the uniform convergence topology. We also denote by $L^2(0,T; X)$ the space of the functions from $[0,T]$ into X which are square integrable, with norm $(\int_0^T \|u(t,\cdot)\|_X^2 \, \mathrm{d}t)^{1/2}$. The following results are proven, respectively, in Lions and Magenes [7, Chapter 1] and Lions [6, Chapter 1]:

Lemma 1.1. *Let $k_1 \geqslant k_2 \geqslant 0$, $k := (k_1 + k_2)/2$, and set*

$$W_{k_1,k_2}(T) := \left\{ u \in L^2(0,T; H^{k_1}) \mid u_t \in L^2(0,T; H^{k_2}) \right\}. \tag{1.12}$$

Then, the injection $W_{k_1,k_2}(T) \hookrightarrow C([0,T]; H^k)$ is continuous; *moreover, there is $C > 0$, independent of T, such that for all $u \in W_{k_1,k_2}(T)$,*

$$\max_{0 \leqslant t \leqslant T} \|u(t)\|_k^2 \leqslant C \left(\int_0^T \|u\|_{k_1}^2 \, \mathrm{d}t \right)^{1/2} \left(\int_0^T \|u_t\|_{k_2}^2 \, \mathrm{d}t \right)^{1/2} + \frac{C}{T} \int_0^T \|u\|_{k_2}^2 \, \mathrm{d}t. \tag{1.13}$$

Likewise, we denote by $C_{\mathrm{b}}([0,+\infty[;X)$ the space of the functions from $[0,+\infty[$ into X which are continuous and bounded, endowed with the uniform convergence topology. A straightforward modification of Lemma 1.1 then holds, with (1.13) replaced by

$$\sup_{t\geqslant 0}\big\|u(t)\big\|_k^2\leqslant C\left(\int_0^{+\infty}\|u\|_{k_1}^2\,\mathrm{d}t\right)^{1/2}\left(\int_0^{+\infty}\|u_t\|_{k_2}^2\,\mathrm{d}t\right)^{1/2}. \tag{1.14}$$

Lemma 1.2. *Let $T>0$, $k_1>k\geqslant k_2\geqslant 0$, and $W_{k_1,k_2}(T)$ as in* (1.12). *Since V is compact, the injection $W_{k_1,k_2}(T)\hookrightarrow L^2(0,T;H^k)$ is compact.*

Finally, for any integer $k\geqslant 0$ and $T>0$, we introduce the Banach spaces

$$X_k(T):=\big\{u\in L^2(0,T;H^{2m+k})\mid u_t\in L^2(0,T;H^k)\big\}, \tag{1.15}$$

$$Y_k(T):=L^2(0,T;H^{2m+k})\cap L^\infty(0,T;H^{m+k}), \tag{1.16}$$

endowed with their natural norms, defined by

$$\|u\|_{X_k(T)}^2:=\int_0^T\big(\|u\|_{2m+k}^2+\|u_t\|_k^2\big)\,\mathrm{d}t, \tag{1.17}$$

$$\|u\|_{Y_k(T)}^2:=\int_0^T\|u\|_{2m+k}^2\,\mathrm{d}t+\sup_{0\leqslant t\leqslant T}\big\|u(t)\big\|_{m+k}^2; \tag{1.18}$$

we also set

$$\sigma_{k,T}(u):=\sup_{0\leqslant t\leqslant T}\big\|u(t)\big\|_{m+k}. \tag{1.19}$$

Note that, by Lemma 1.1, $X_k(T)\hookrightarrow Y_k(T)$, and, for all $u\in X_k(T)$,

$$\sigma_{k,T}(u)\leqslant C\,\|u\|_{X_k(T)}, \tag{1.20}$$

with C independent of u. We extend these definitions in the natural way when $T=+\infty$.

1.4. Properties of N and M

From (1.1) we deduce that, if $u_1,\dots,u_{m+1}\in C^\infty(V)$, the quantity $N(u_1,\dots,u_m)$ is completely symmetric in all its arguments. Using integration by parts and recalling that the metric g is Kähler, we see that the same is true for

$$I(u_1,\dots,u_{m+1}):=\big\langle N(u_1,\dots,u_m),u_{m+1}\big\rangle. \tag{1.21}$$

Moreover, by Hölder's inequality, (1.10) and (1.11),

$$\begin{aligned}\big|N(u_1,\dots,u_m)\big|_2&\leqslant C_H\prod_{k=1}^m|\nabla^2u_k|_{2m}\leqslant C_HC_S\prod_{k=1}^m\|u_k\|_{m+1}\\&\leqslant C_HC_SC_I\prod_{k=1}^m\|u_k\|_m^{1-1/m}\|u_k\|_{2m}^{1/m}.\end{aligned} \tag{1.22}$$

Likewise, if $p_1, \dots p_{m+1} \in [1, +\infty]$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_{m+1}} = 1$,

$$\left|I(u_1, \dots, u_{m+1})\right| \leqslant C_H \left(\prod_{k=1}^{m} |\nabla^2 u_k|_{p_k}\right) |u_{m+1}|_{p_{m+1}}, \tag{1.23}$$

with C_H depending only on m. In fact, as we see by integration by parts, I also satisfies the estimate

$$\left|I(u_1, \dots, u_{m+1})\right| \leqslant C_H \prod_{k=1}^{m-1} |\nabla^2 u_k|_{p_k} \prod_{k=m}^{m+1} |\nabla u_k|_{p_k}. \tag{1.24}$$

In particular, taking $p_1 = \cdots = p_{m-1} = m$ and $p_m = p_{m+1} = 2m$, by (1.24) and (1.10)

$$\left|I(u_1, \dots, u_{m+1})\right| \leqslant C_H C_S \prod_{k=1}^{m+1} \|u_k\|_m. \tag{1.25}$$

Finally, taking into account that, since g is Kähler, $\nabla_{\lambda\alpha\bar\beta} v = \nabla_{\alpha\bar\beta\lambda} v$ for any function v, we have that

$$\nabla_\lambda N(u_1, \dots, u_m) = \sum_{k=1}^{m} N(u_1, \dots, \nabla_\lambda u_k, \dots, u_m), \tag{1.26}$$

from which it follows that

$$I(u_1, \dots, u_m, \Delta u_{m+1}) = \sum_{k=1}^{m} \left\langle N(u_1, \dots, \nabla^\lambda u_k, \dots, u_m), \nabla_\lambda u_{m+1}\right\rangle. \tag{1.27}$$

We now turn to Eq. (1.5), and claim:

Lemma 1.3. *Let $h \geqslant 0$, and $u \in H^{m+h}$. There is a unique $f \in H^{m+h}$, solution of* (1.5). *Moreover, there exists C, independent of u, such that*

$$\|f\|_{m+h} \leqslant C \|u\|_m^{m-1} \|u\|_{m+h}. \tag{1.28}$$

Proof. By (1.25), the inequality

$$\left|\left\langle M(u), \varphi\right\rangle\right| \leqslant C \|u\|_m^m \|\varphi\|_m \tag{1.29}$$

holds for all $\varphi \in H^m$, with C independent of u and φ. Hence, by Riesz' representation theorem, there exists a unique $f \in H^m$, such that

$$\langle f, \varphi\rangle_{H^m} = -\left\langle M(u), \varphi\right\rangle$$

for all $\varphi \in H^m$. Thus, (1.5) holds in distributional sense. Taking $\varphi = f$ and using (1.29), we obtain (1.28) when $h = 0$. If $h \geqslant 1$, let $k := \frac{h}{2}$. Applying Δ^k to (1.5), then multiplying by $\Delta^k f$ and integrating on V, we obtain

$$\begin{aligned}\|f\|_{m+2k}^2 &= \langle \Delta^{k+m} f, \Delta^k f\rangle = -\left\langle \Delta^k M(u), \Delta^k f\right\rangle \\ &= (-1)^{k+1} \sum \left(I(\nabla^{q_1} u, \dots, \nabla^{q_m} u, \Delta^k f) + J_{q_1,\dots,q_m}(u)\right),\end{aligned} \tag{1.30}$$

where the sum is taken over multiindices of orders $q_1, \dots, q_m$ such that $q_1 + \cdots + q_m = 2k = h$, and $J_{q_1,\dots,q_m}(u)$ has the following structure. By permutations of covariant derivatives, and induction, we see that, for all $p \geqslant 1$, the difference

$$\nabla_{a_1 \dots a_p \alpha\bar\beta} u - \nabla_{\alpha\bar\beta a_1 \dots a_p} u, \quad a_1 \dots a_p \in \{1, \dots, m, \bar 1, \dots, \bar m\},$$

can be written as a sum of tensor products of covariant derivatives of the function u and the curvature tensor, of order $\leqslant p$. Hence,

$$J_{q_1,\dots,q_m}(u) = \int_V E_{q_1,\dots,q_m}(u)\,\mathrm{d}x,$$

where $E_{q_1,\dots,q_m}(u)$ is a linear combination of terms like $\nabla^{\tilde{q}_1}u\cdots\nabla^{\tilde{q}_m}u\,\Delta^k f$, with $0 \leqslant \tilde{q}_j \leqslant q_j$. It follows that to estimate $J_{q_1,\dots,q_m}(u)$, we have to bound terms like

$$\int_V h_{q_1,\dots,q_m}\nabla^{q_1}u\cdots\nabla^{q_m}u\,\Delta^k f\,\mathrm{d}x.$$

That is, recalling (1.25), the interpolation inequality (1.11), and that $q_1 + \cdots + q_m = 2k = h$,

$$\begin{aligned}
\bigl|I(\nabla^{q_1}u,\dots,\nabla^{q_m}u,\Delta^k f)\bigr| &\leqslant C_S\Biggl(\prod_{j=1}^m \|\nabla^{q_j}u\|_m\Biggr)\|\Delta^k f\|_m \\
&\leqslant C_S\Biggl(\prod_{j=1}^m \|u\|_{m+q_j}\Biggr)\|f\|_{m+2k} \\
&\leqslant C_S C_I\Biggl(\prod_{j=1}^m \|u\|_m^{1-q_j/2k}\|u\|_{m+2k}^{q_j/2k}\Biggr)\|f\|_{m+2k} \\
&= C_S C_I\|u\|_m^{m-1}\|u\|_{m+h}\|f\|_{m+h}. \qquad (1.31)
\end{aligned}$$

The term $J_{q_1,\dots,q_m}(u)$ can be estimated in the same way; hence, (1.28) follows from (1.30) and (1.31). This completes the proof of Lemma 1.3. □

Corollary 1.1. *For all $u \in L^2(0,T;H^{2m}) \cap L^\infty(0,T;H^m)$, there is a unique $f \in L^2(0,T;H^{2m}) \cap L^\infty(0,T;H^m)$, which satisfies* (1.5) *for almost all $t \in [0,T]$, and the estimates*

$$\int_0^T \|f\|_{2m}^2\,\mathrm{d}t \leqslant C\sup_{0\leqslant t\leqslant T}\bigl\|u(t)\bigr\|_m^{2(m-1)}\int_0^T \|u\|_{2m}^2\,\mathrm{d}t, \qquad (1.32)$$

$$\sup_{0\leqslant t\leqslant T}\bigl\|f(t)\bigr\|_m \leqslant C\sup_{0\leqslant t\leqslant T}\bigl\|u(t)\bigr\|_m^m. \qquad (1.33)$$

Proof. (1.32) follows from (1.28), with $h = m$, after integration on $[0,T]$. Likewise, (1.33) follows from (1.28), with $h = 0$. □

2. Well posedness

In this section we prove that if problem (P) has a solution $u \in X_0(T)$ (the space defined in (1.15)), for some $T > 0$, then u depends continuously on the data $\{\varphi, u_0\}$.

Theorem 2.1. *Let $T > 0$, $u_0, \tilde{u}_0 \in H^m$, and $\varphi, \tilde{\varphi} \in Y_0(T)$. Assume that problem* (P) *has solutions $\{u, f\}, \{\tilde{u}, \tilde{f}\} \in X_0(T) \times Y_0(T)$, corresponding respectively to the data $\{\varphi, u_0\}$ and $\{\tilde{\varphi}, \tilde{u}_0\}$. Then, the difference $u - \tilde{u}$ satisfies the estimate*

$$\|u-\tilde{u}\|_{X_0(T)}^2 + \|f-\tilde{f}\|_{Y_0(T)}^2 \leqslant C_*\bigl(\|u_0-\tilde{u}_0\|_m^2 + Q(\varphi-\tilde{\varphi})\bigr), \qquad (2.1)$$

where, for $\chi \in Y_0(T)$*,*

$$Q(\chi) := \Big(\sup_{0\leqslant t\leqslant T} \|\chi(t)\|_m^2\Big)^{1-1/m} \left(\int_0^T \|\chi\|_{2m}^2 \, \mathrm{d}t\right)^{1/m} \tag{2.2}$$

(*which is quadratic in* χ*), and* C_* *depends on* u*,* $\tilde{u}$*,* f*,* $\tilde{f}$*,* φ *and* $\tilde{\varphi}$*. In particular, there is at most one solution of problem* (P) *in* $X_0(T)$*.*

Proof. Let $z := u - \tilde{u}$, $g := f - \tilde{f}$, and $\psi := \varphi - \tilde{\varphi}$. By difference, and the symmetry of N, z and g solve the system

$$z_t + \Delta^m z = A + B, \tag{2.3}$$

$$\Delta^m g = -C, \tag{2.4}$$

with $A = A(g, z, u, \tilde{u}, \tilde{f})$, $B = B(z, \psi, \tilde{u}, \varphi, \tilde{\varphi})$ and $C = C(z, u, \tilde{u})$ defined by

$$A := N(g, u^{(m-1)}) + \sum_{k=2}^m N(\tilde{f}, z, u^{(m-k)}, \tilde{u}^{(k-2)}) =: \sum_{k=1}^m A_k, \tag{2.5}$$

$$B := N(z, \varphi^{(m-1)}) + \sum_{k=2}^m N(\tilde{u}, \psi, \varphi^{(m-k)}, \tilde{\varphi}^{(k-2)}) =: \sum_{k=1}^m B_k, \tag{2.6}$$

$$C := \sum_{k=1}^m N(z, u^{(m-k)}, \tilde{u}^{(k-1)}) =: \sum_{k=1}^m C_k. \tag{2.7}$$

We multiply (2.3) in H^0 by $\Delta^m z$ and z_t, and (2.4) by $\Delta^m g$; adding the resulting identities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_m^2 + \|z\|_{2m}^2 + \|z_t\|_0^2 + \|g\|_{2m}^2 = \langle A + B, \Delta^m z + z_t\rangle - \langle C, \Delta^m g\rangle. \tag{2.8}$$

Recalling (1.22), we start the estimate of A with

$$\begin{aligned}
\langle A_1, \Delta^m z + z_t\rangle &\leqslant C\|g\|_m^{\frac{m-1}{m}} \|g\|_{2m}^{\frac{1}{m}} \|u\|_m^{\frac{(m-1)^2}{m}} \|u\|_{2m}^{\frac{m-1}{m}} \big(\|z\|_{2m} + \|z_t\|_0\big) \\
&\leqslant \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + C_\eta \|g\|_m^{\frac{2(m-1)}{m}} \|g\|_{2m}^{\frac{2}{m}} \|u\|_m^{\frac{2(m-1)^2}{m}} \|u\|_{2m}^{\frac{2(m-1)}{m}} \\
&\leqslant \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + \eta'\|g\|_{2m}^2 + C_{\eta\eta'}\|g\|_m^2 \|u\|_m^{2(m-1)} \|u\|_{2m}^2,
\end{aligned} \tag{2.9}$$

having used the inequality

$$ab \leqslant \sigma a^m + C_\sigma b^{m/(m-1)}, \quad a, b \geqslant 0,\ m > 1,\ \sigma > 0.$$

To estimate $\|g\|_m$, we first multiply (2.4) in H^0 by g, to obtain

$$\|g\|_m^2 = \langle \Delta^m g, g\rangle = -\sum_{k=1}^m \langle C_k, g\rangle; \tag{2.10}$$

then, we simplify notations, denoting by $\hat{u}$ either one of the functions u or $\tilde{u}$, and by $\hat{u}^{(m-1)}$ a generic product (in N) of k_1 factors u and k_2 factors $\tilde{u}$, with $k_1 + k_2 = m - 1$. Thus, by (1.25),

$$\langle C_k, g\rangle = \big\langle N(z, \hat{u}^{(m-1)}), g\big\rangle \leqslant CC_S\|z\|_m \|\hat{u}\|_m^{m-1}\|g\|_m, \tag{2.11}$$

so that, from (2.10),

$$\|g\|_m \leqslant C \sum\nolimits_* \|z\|_m \|\hat{u}\|_m^{m-1}, \tag{2.12}$$

where $\sum_*$ refers to all possible combinations of $\hat{u} \in \{u, \tilde{u}\}$. By (2.12), we deduce from (2.9) (omitting the $\sum_*$ for simplicity)

$$\begin{aligned}\langle A_1, \Delta^m z + z_t\rangle \leqslant{}& \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + \eta'\|g\|_{2m}^2 \\ &+ C_{\eta\eta'}\|\hat{u}\|_m^{2(m-1)}\|u\|_m^{2(m-1)}\|u\|_{2m}^2\|z\|_m^2.\end{aligned} \tag{2.13}$$

The estimate of the other terms of $\langle A, \Delta^m z + z_t\rangle$ is analogous. Because of the symmetry of N, and recalling estimate (1.28) for $\tilde{f}$, if $k \geqslant 2$,

$$\begin{aligned}\langle A_k, \Delta^m z + z_t\rangle &\leqslant C\|\hat{u}\|_m^{\frac{2(m-1)^2}{m}}\|\hat{u}\|_{2m}^{\frac{m-2}{m}}\|\tilde{f}\|_{2m}^{\frac{1}{m}}\|z\|_m^{\frac{m-1}{m}}\|z\|_{2m}^{\frac{1}{m}}\big(\|z\|_{2m} + \|z_t\|_0\big) \\ &\leqslant \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + C_\eta\|\hat{u}\|_m^{4(m-1)}\|\hat{u}\|_{2m}^{\frac{2(m-2)}{m-1}}\|\tilde{f}\|_{2m}^{\frac{2}{m-1}}\|z\|_m^2 \\ &\leqslant \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + C_\eta\|\hat{u}\|_m^{4(m-1)}\big(\|\hat{u}\|_{2m}^2 + \|\tilde{f}\|_{2m}^2\big)\|z\|_m^2.\end{aligned} \tag{2.14}$$

Likewise,

$$\begin{aligned}\langle B, \Delta^m z + z_t\rangle \leqslant{}& \eta\big(\|z\|_{2m}^2 + \|z_t\|_0^2\big) + C_\eta\|\varphi\|_m^{2(m-1)}\|\varphi\|_{2m}^2\|z\|_m^2 \\ &+ C_\eta\|\tilde{u}\|_m^{\frac{2(m-1)}{m}}\|\hat{\varphi}\|_m^{\frac{2(m-2)(m-1)}{m}}\|\psi\|_m^{\frac{2(m-1)}{m}}\|\tilde{u}\|_{2m}^{\frac{2}{m}}\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\|\psi\|_{2m}^{\frac{2}{m}},\end{aligned} \tag{2.15}$$

and

$$\langle C, \Delta^m g\rangle \leqslant \eta\|g\|_{2m}^2 + \eta'\|z\|_{2m}^2 + C_{\eta\eta'}\|\tilde{u}\|_m^{2(m-1)}\|\tilde{u}\|_{2m}^2\|z\|_m^2. \tag{2.16}$$

We now note that $\hat{u}$, $\tilde{f}$ and $\hat{\varphi} \in L^\infty(0, T; H^m)$: $\hat{u}$ because u and $\tilde{u} \in X_0(T)$; $\tilde{f}$ by Lemma 1.3, and $\hat{\varphi}$ by assumption. We abbreviate $\sigma_{0,T}(\hat{u}) =: \sigma(\hat{u})$ (recall (1.19)), and analogously for $\sigma(\tilde{f})$, $\sigma(\varphi)$, $\sigma(\hat{\varphi})$, $\sigma(\psi)$; inserting (2.13), ..., (2.16) into (2.8), and taking η and η' sufficiently small, we obtain

$$\begin{aligned}2\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_m^2 + \|z\|_{2m}^2 + \|z_t\|_0^2 + \|g\|_{2m}^2 \leqslant{}& \sum\nolimits_* D(\hat{u}, \hat{\varphi}, \psi)\big(\|\hat{u}\|_{2m}^{\frac{2}{m}}\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\|\psi\|_{2m}^{\frac{2}{m}}\big) \\ &+ \sum\nolimits_* E(\hat{u}, \varphi)\big(\|\hat{u}\|_{2m}^2 + \|\tilde{f}\|_{2m}^2 + \|\varphi\|_{2m}^2\big)\|z\|_m^2,\end{aligned} \tag{2.17}$$

where

$$D(\hat{u}, \hat{\varphi}, \psi) := C\big(\sigma(\hat{u})\big)^{\frac{2(m-1)}{m}}\big(\sigma(\hat{\varphi})\big)^{\frac{2(m-2)(m-1)}{m}}\big(\sigma(\psi)\big)^{\frac{2(m-1)}{m}}, \tag{2.18}$$

$$E(\hat{u}, \hat{\varphi}) := C\big(1 + \big(\sigma(\hat{u})\big)^{4(m-1)} + \big(\sigma(\varphi)\big)^{2(m-1)}\big). \tag{2.19}$$

Integrating (2.17) on $[0, t]$, with $0 < t \leqslant T$, by Gronwall's inequality,

$$\begin{aligned}&\big\|z(t)\big\|_m^2 + \int_0^t \big(\|z\|_{2m}^2 + \|z_t\|_0^2 + \|g\|_{2m}^2\big)\,\mathrm{d}\theta \\ &\quad\leqslant \left(\big\|z(0)\big\|_m^2 + \sum\nolimits_* D(\hat{u}, \hat{\varphi}, \psi)\int_0^T \|\hat{u}\|_{2m}^{\frac{2}{m}}\|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}}\|\psi\|_{2m}^{\frac{2}{m}}\,\mathrm{d}t\right) \\ &\qquad\times\exp\left(\sum\nolimits_* E(\hat{u}, \varphi)\left(\int_0^T \|\hat{u}\|_{2m}^2 + \|\tilde{f}\|_{2m}^2 + \|\varphi\|_{2m}^2\right)\mathrm{d}t\right).\end{aligned} \tag{2.20}$$

By Hölder's inequality,

$$\int_0^T \left(\|\hat{u}\|_{2m}^{\frac{2}{m}} \|\hat{\varphi}\|_{2m}^{\frac{2(m-2)}{m}} \|\psi\|_{2m}^{\frac{2}{m}}\right) \mathrm{d}t \leqslant \left(\int_0^T \|\hat{u}\|_{2m}^2 \,\mathrm{d}t\right)^{\frac{1}{m}} \left(\int_0^T \|\hat{\varphi}\|_{2m}^2 \,\mathrm{d}t\right)^{\frac{m-2}{m}} \left(\int_0^T \|\psi\|_{2m}^2 \,\mathrm{d}t\right)^{\frac{1}{m}}; \tag{2.21}$$

recalling then the definition (2.18) of $D(\hat{u}, \hat{\varphi}, \psi)$, (2.21) and (2.20) imply

$$\left\|z(t)\right\|_m^2 + \int_0^t \left(\|z\|_{2m}^2 + \|z_t\|_0^2 + \|g\|_{2m}^2\right) \mathrm{d}\theta \leqslant C_* \left(\left\|z(0)\right\|_m^2 + \left(\sup_{0\leqslant t\leqslant T} \left\|\psi(t)\right\|_m^2\right)^{1-1/m} \left(\int_0^T \|\psi\|_{2m}^2 \,\mathrm{d}t\right)^{1/m}\right), \tag{2.22}$$

with C_* as in the statement of Theorem 2.1. To conclude, we estimate $\|f(t) - \tilde{f}(t)\|_m$ by inserting (2.22) into (2.12); adding the resulting estimate to (2.22), we obtain (2.1), and the proof of Theorem 2.1 is complete. □

We remark that Theorem 2.1 can be extended, in a straightforward manner, to the case $T = +\infty$.

3. Local existence

In this section we prove that problem (P) has a local solution $u \in X_1(\tau)$, for some $\tau > 0$; the method we follow cannot be applied to obtain local existence in $X_0(\tau)$, unless the data $\{u_0, \varphi\}$ are sufficiently small (see Theorem 4.1 below).

3.1. The main result of this section is

Theorem 3.1. *Let $T > 0$, $u_0 \in H^{m+1}$, and $\varphi \in Y_1(T)$. Set*

$$Q_1(\varphi) := \left(\sigma_{1,T}(\varphi)\right)^{2(m-1-1/m)} \left(\int_0^T \|\varphi\|_{2m+1}^2 \,\mathrm{d}t\right)^{1/m} + \left(\sigma_{1,T}(\varphi)\right)^{2m} \tag{3.1}$$

(*with $\sigma_{1,T}$ as in* (1.19)), *and*

$$R := \max\left\{2\sqrt{2}\|u_0\|_{m+1} \mathrm{e}^{Q_1(\varphi)}, 1\right\}. \tag{3.2}$$

There exists $\tau_* \in \,]0, T]$, and a unique pair $\{u, f\} \in X_1(\tau_*) \times Y_1(\tau_*)$, solution of problem (P). *Moreover, u satisfies the estimate*

$$\sup_{0\leqslant t\leqslant \tau_*} \left\|u(t)\right\|_{m+1}^2 + \int_0^{\tau_*} \left(\|u\|_{2m+1}^2 + \|u_t\|_1^2\right) \mathrm{d}t \leqslant R^2. \tag{3.3}$$

Proof. We resort to a linearization and fixed point technique, as e.g. in Kato [5]. Given any $\tau \in]0, T]$, and R as in (3.2), recalling definitions (1.17) and (1.19), we set

$$B_1(\tau) := \{u \in X_1(\tau) \mid (\sigma_{1,\tau}(u))^2 + \|u\|^2_{X_1(\tau)} \leqslant R^2\}. \tag{3.4}$$

For fixed $w \in B_1(\tau)$, we consider the linearized problem

$$u_t + \Delta^m u = N(f, w^{(m-2)}, u) + N(\varphi^{(m-1)}, u), \tag{3.5}$$

$$\Delta^m f = -M(w), \tag{3.6}$$

$$u(0, \cdot) = u_0, \tag{3.7}$$

which we refer to as "problem (LP)". A solution to this linear problem can be found in several ways; for example, resorting to a Galerkin approximation scheme, based on a priori estimates similar to those we establish in the proof of Theorem 3.3 below. In particular, we have

Theorem 3.2. *Let $T > 0$, $u_0 \in H^{m+1}$, and $\varphi \in Y_1(T)$. For all $\tau \in]0, T]$ and $w \in X_1(\tau)$, problem* (LP) *has a unique solution $\{u, f\} \in X_1(\tau) \times Y_1(\tau)$.*

Theorem 3.2 allows us to define a map $\Phi : X_1(\tau) \to X_1(\tau)$, by

$$w \mapsto u =: \Phi(w), \tag{3.8}$$

where $\{u, f\}$ is the solution of problem (LP). The next step is fundamental.

Theorem 3.3. *There exists $\tau_* \in]0, T]$, depending only on R of* (3.2)*, such that the map Φ defined in* (3.8) *maps the ball $B_1(\tau_*)$ into itself.*

We prove this theorem in Section 3.2 below. Assuming it true, consider the sequences $(u^n)_{n\geqslant 0}$, $(f^n)_{n\geqslant 0}$, defined iteratively by

$$u^0 := u_0, \quad u^{n+1} := \Phi(u^n), \quad \Delta^m f^n = -M(u^n) \tag{3.9}$$

(the latter as per (3.6)). By Theorem 3.3, $(u^n)_{n\geqslant 0}$ lies in a bounded set of $L^2(0, \tau_*; H^{2m+1})$, with $(u^n_t)_{n\geqslant 0}$ in a bounded set of $L^2(0, \tau_*; H^1)$. Hence, there is a subsequence, still denoted $(u^n)_{n\geqslant 0}$, and a function $u \in B_1(\tau_*)$, such that

$$u^n \to u \quad \text{weakly in } L^2(0, \tau_*; H^{2m+1}), \tag{3.10}$$

$$u^n_t \to u_t \quad \text{weakly in } L^2(0, \tau_*; H^1). \tag{3.11}$$

In fact, the whole sequence converges, because of the uniqueness result proven in Theorem 2.1. Now, since V is compact, by Lemma 1.2, the injection

$$X_1(\tau_*) \hookrightarrow L^2(0, \tau_*; H^{2m}) \tag{3.12}$$

is compact; hence, from (3.10) and (3.11), we deduce that

$$u^n \to u \quad \text{in } L^2(0, \tau_*; H^{2m}) \tag{3.13}$$

(taking a further subsequence if necessary). By (1.13), (3.13) and (3.11) imply that

$$u^n \to u \quad \text{in } C([0, \tau_*]; H^m). \tag{3.14}$$

We now show that

$$M(u^n) \to M(u) \quad \text{in } L^2(0, \tau_*; H^0). \tag{3.15}$$

Indeed, setting $z^n := u^n - u$ we compute, as in (2.7),

$$M(u^n) - M(u) = \sum_{k=1}^{m} N\big(u^{(k-1)}, z^n, (u^n)^{(m-k)}\big) =: \sum_{k=1}^{m} D_k. \tag{3.16}$$

Recalling (1.22), and that u^n and $u \in B_1(\tau_*)$ (recall (3.4)), we estimate

$$|D_k|_2 \leqslant C_H C_S \|u\|_{m+1}^{k-1} \|z^n\|_{m+1} \|u^n\|_{m+1}^{m-k} \leqslant C_H C_S R^{m-1} \|z^n\|_{m+1}. \tag{3.17}$$

Thus, (3.15) follows from (3.13), which implies that $u^n \to u$ also in $L^2(0, \tau_*; H^{m+1})$.

Define now $f \in Y_1(\tau_*)$ by (1.5), that is, $\Delta^m f = -M(u)$, where u is as in (3.10). Then, since

$$\Delta^m (f^n - f) = M(u) - M(u^n), \tag{3.18}$$

from (3.15) it follows that

$$f^n \to f \quad \text{in } L^2(0, \tau_*; H^{2m}). \tag{3.19}$$

Finally, we show that

$$N\big(f^n, (u^n)^{(m-2)}, u^{n+1}\big) \to N(f, u^{(m-1)}) \quad \text{in } L^2(0, \tau_*; H^0). \tag{3.20}$$

Indeed, we compute that

$$N\big(f^n, (u^n)^{(m-2)}, u^{n+1}\big) - N(f, u^{(m-1)}) = \underbrace{N\big(f^n - f, (u^n)^{(m-2)}, u^{n+1}\big)}_{=:N_n^{(1)}}$$

$$+ \underbrace{\sum_{k=2}^{m-1} N\big(f, u^{(k-2)}, z^n, (u^n)^{(m-k-1)}, u^{n+1}\big)}_{=:N_n^{(2)}} + \underbrace{N(f, u^{(m-2)}, z^{n+1})}_{=:N_n^{(3)}}. \tag{3.21}$$

By (1.22), and since $u^n, u^{n+1} \in B_1(\tau_*)$,

$$\begin{aligned} \|N_n^{(1)}\|_0 &\leqslant C_H C_S \|f^n - f\|_{m+1} \|u^n\|_{m+1}^{m-2} \|u^{n+1}\|_{m+1} \\ &\leqslant C_H C_S R^{m-1} \|f^n - f\|_{m+1}; \end{aligned} \tag{3.22}$$

consequently, $N_n^{(1)} \to 0$ in $L^2(0, \tau_*; H^0)$, because of (3.19). Acting likewise, we see that

$$\|N_n^{(2)}\|_0 \leqslant C_H C_S R^{m-2} \|f\|_{m+1} \|z^n\|_{m+1}, \tag{3.23}$$

$$\|N_n^{(3)}\|_0 \leqslant C_H C_S R^{m-2} \|f\|_{m+1} \|z^{n+1}\|_{m+1}; \tag{3.24}$$

since $f \in Y_1(\tau_*) \hookrightarrow L^\infty(0, \tau_*; H^{m+1})$, (3.13) implies that $N_n^{(2)} \to 0$ and $N_n^{(3)} \to 0$ in $L^2(0, \tau_*; H^0)$. Hence, (3.20) follows, by (3.21). From (3.13) and (3.20), it follows that also

$$\begin{aligned} u_t^{n+1} &= -\Delta^m u^{n+1} + N\big(f^n, (u^n)^{(m-2)}, u^{n+1}\big) + N(\varphi^{(m-1)}, u^{n+1}) \\ &\to -\Delta^m u + N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u) \end{aligned} \tag{3.25}$$

strongly in $L^2(0, \tau_*; H^0)$. Since also $u_t^n \to u_t$ weakly in $L^2(0, \tau_*; H^1)$ (by (3.11)), we deduce from (3.25) that u satisfies Eq. (1.4). But then, (3.25) implies that $u_t^n \to u_t$ in $L^2(0, \tau_*; H^0)$; together with (3.13), this means that $u^n \to u$ in $X_0(\tau_*)$, with respect to the norm defined in (1.17). Moreover, u satisfies the initial condition (1.6), because, by (3.14), $u^n(0) = u_0 \to u(0)$ in H^m. Eq. (1.5) follows from (3.19) and (3.15); finally, estimate (3.3) follows from the fact that $u \in B_1(\tau_*)$. The proof of Theorem 3.1 is thus complete. □

3.2. Proof of Theorem 3.3

Let $\tau \in]0, T]$, fix $w \in B_1(\tau)$, and multiply Eq. (3.6) in H^0 by $\Delta^{m+1}u + \Delta u_t$. This procedure is somewhat formal, since we do not know that $\Delta^{m+1}u + \Delta u_t \in H^0$; however, we can establish our estimates for the Galerkin approximations of u, which we would use to solve problem (LP) and are smooth, and then recover the desired estimates for u itself. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{m+1}^2 + \|u\|_{2m+1}^2 + \|u_t\|_1^2 = A + B, \tag{3.26}$$

where

$$A := \langle N(f, w^{(m-2)}, u), \Delta^{m+1}u + \Delta u_t\rangle, \tag{3.27}$$

$$B := \langle N(\varphi^{(m-1)}, u), \Delta^{m+1}u + \Delta u_t\rangle. \tag{3.28}$$

Integrating by parts, and recalling (1.27),

$$\begin{aligned} A = {} & \underbrace{\langle N(\nabla f, w^{(m-2)}, u), \nabla(\Delta^m u + u_t)\rangle}_{=:A_1} \\ & + (m-2)\underbrace{\langle N(f, w^{(m-3)}, \nabla w, u), \nabla(\Delta^m u + u_t)\rangle}_{=:A_2} \\ & + \underbrace{\langle N(f, w^{(m-2)}, \nabla u), \nabla(\Delta^m u + u_t)\rangle}_{=:A_3}. \end{aligned} \tag{3.29}$$

We estimate the terms of A as before, starting with

$$\begin{aligned} |A_1| &\leqslant C_H|\nabla^3 f|_{2m}|\nabla^2 w|_{2m}^{m-2}|\nabla^2 u|_{2m}\big|\nabla(\Delta^m u + u_t)\big|_2 \\ &\leqslant C_H C_S\|f\|_{m+2}\|w\|_{m+1}^{m-2}\|u\|_{m+1}\big(\|u\|_{2m+1} + \|u_t\|_1\big). \end{aligned} \tag{3.30}$$

From (3.6) and (1.22),

$$\|f\|_{m+2}^2 = \langle\Delta^m f, \Delta^2 f\rangle = -\langle M(w), \Delta^2 f\rangle \leqslant CC_S\|w\|_{m+1}^m\|f\|_4;$$

since $m \geqslant 2$, this implies

$$\|f\|_{m+2} \leqslant C\|w\|_{m+1}^m. \tag{3.31}$$

Inserting this into (3.30), and recalling that $\|w\|_{m+1} \leqslant R$ because $w \in B_1(\tau)$, we obtain that for all $\eta > 0$,

$$\begin{aligned} |A_1| &\leqslant CR^{2(m-1)}\|u\|_{m+1}\big(\|u\|_{2m+1} + \|u_t\|_1\big) \\ &\leqslant \eta\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big) + C_\eta R^{4(m-1)}\|u\|_{m+1}^2. \end{aligned} \tag{3.32}$$

Similarly,

$$|A_2| \leqslant C\|f\|_{m+1}\|w\|_{m+1}^{m-3}\|w\|_{m+2}\|u\|_{m+1}\big(\|u\|_{2m+1} + \|u_t\|_1\big). \tag{3.33}$$

As in (3.31),

$$\|f\|_{m+1} \leqslant C\|w\|_{m+1}^m, \tag{3.34}$$

and by (1.11),

$$\|w\|_{m+2} \leqslant C_I\|w\|_{2m+1}^{1/m}\|w\|_{m+1}^{1-1/m}. \tag{3.35}$$

Inserting (3.34) and (3.35) into (3.33),

$$|A_2| \leqslant \eta\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big) + C_\eta R^{4(m-1)-2/m}\|w\|_{2m+1}^{2/m}\|u\|_{m+1}^2. \tag{3.36}$$

We proceed analogously for A_3:

$$\begin{aligned}|A_3| &\leqslant C\|f\|_{m+1}\|w\|_{m+1}^{m-2}\|u\|_{m+2}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant CR^{2(m-1)}\|u\|_{m+1}^{1-1/m}\|u\|_{2m+1}^{1/m}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant \eta\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big) + C_\eta R^{4m}\|u\|_{m+1}^2.\end{aligned} \tag{3.37}$$

We proceed now to estimate B. From (3.28) and (1.27),

$$\begin{aligned}B = (m-1)&\underbrace{\big\langle N(\varphi^{(m-2)}, \nabla\varphi, u), \nabla(\Delta^m u + u_t)\big\rangle}_{=:B_1}\\ &+ \underbrace{\big\langle N(\varphi^{(m-1)}, \nabla u), \nabla(\Delta^m u + u_t)\big\rangle}_{=:B_2}.\end{aligned} \tag{3.38}$$

By (3.35),

$$\begin{aligned}|B_1| &\leqslant C\|\varphi\|_{m+1}^{m-2}\|\varphi\|_{m+2}\|u\|_{m+1}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant C\|\varphi\|_{m+1}^{m-1-1/m}\|\varphi\|_{2m+1}^{1/m}\|u\|_{m+1}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant \eta\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big) + C_\eta\|\varphi\|_{m+1}^{2(m-1-1/m)}\|\varphi\|_{2m+1}^{2/m}\|u\|_{m+1}^2,\end{aligned} \tag{3.39}$$

$$\begin{aligned}|B_2| &\leqslant C\|\varphi\|_{m+1}^{m-1}\|u\|_{m+2}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant C\|\varphi\|_{m+1}^{m-1}\|u\|_{m+1}^{1-1/m}\|u\|_{2m+1}^{1/m}\big(\|u\|_{2m+1} + \|u_t\|_1\big)\\ &\leqslant \eta\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big) + C_\eta\|\varphi\|_{m+1}^{2m}\|u\|_{m+1}^2.\end{aligned} \tag{3.40}$$

Choosing then η sufficiently small, and recalling that $R \geqslant 1$, the above established bounds on A_1, A_2, A_3, B_1 and B_2, allow us to deduce from (3.26) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{m+1}^2 + \frac{1}{2}\big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big)\\ &\quad\leqslant C\big(R^{4m} + R^{4(m-1)-2/m}\|w\|_{2m+1}^{2/m} + \big(\sigma_{1,T}(\varphi)\big)^{2m}\\ &\qquad + \big(\sigma_{1,T}(\varphi)\big)^{2(m-1-1/m)}\|\varphi\|_{2m+1}^{2/m}\big)\|u\|_{m+1}^2.\end{aligned} \tag{3.41}$$

Assume then that $\tau \leqslant 1$, and integrate (3.41) on $[0, \tau]$; by Gronwall's and Hölder's inequalities, we obtain

$$\big\|u(t)\big\|_{m+1}^2 + \int_0^t \big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big)\,\mathrm{d}t \leqslant 2\|u_0\|_{m+1}^2 \exp(C\tau^{1-1/m}\Lambda), \tag{3.42}$$

where, since $w \in B_1(\tau)$,

$$\begin{aligned}\Lambda &:= R^{4m} + R^{4(m-1)-2/m}\left(\int_0^\tau \|w\|_{2m+1}^2\,\mathrm{d}t\right)^{1/m} + \big(\sigma_{1,T}(\varphi)\big)^{2m} \\ &\quad + \big(\sigma_{1,T}(\varphi)\big)^{2(m-1-1/m)}\left(\int_0^\tau \|\varphi\|_{2m+1}^2\,\mathrm{d}t\right)^{1/m} \\ &\leqslant R^{4m} + R^{4(m-1)} + Q_1(\varphi), \end{aligned} \tag{3.43}$$

in accord to definition (3.1) of $Q_1(\varphi)$. Choosing then $\tau_* \leqslant 1$ such that also

$$C\tau_*^{1-1/m} \leqslant \min\left\{1, \frac{\ln 2}{2R^{4m}}\right\}, \tag{3.44}$$

we deduce from (3.42), (3.43) and (3.2), that for all $t \in]0, \tau_*]$,

$$\begin{aligned}\big\|u(t)\big\|_{m+1}^2 + \int_0^t \big(\|u\|_{2m+1}^2 + \|u_t\|_1^2\big)\,\mathrm{d}t &\leqslant 2\|u_0\|_{m+1}^2 \mathrm{e}^{Q_1(\varphi)} \exp(2C\tau_*^{1-1/m}R^{4m}) \\ &\leqslant 4\|u_0\|_{m+1}^2 \mathrm{e}^{Q_1(\varphi)} \leqslant \frac{1}{2}R^2. \end{aligned} \tag{3.45}$$

This proves that $u \in B_1(\tau_*)$, as claimed. This completes the proof of Theorem 3.3. □

4. Regularity

Regularity results for problem (P) are not difficult to establish; indeed, Theorem 3.1 can be easily generalized, with analogous proof, into

Theorem 4.1. *Let $k \geqslant 0$, $T > 0$, $u_0 \in H^{m+k+1}$, and $\varphi \in Y_{k+1}(T)$. There exists $\tau_{k+1} \in]0, T]$, such that problem* (P) *has a unique solution $u \in X_{k+1}(\tau_{k+1})$.*

Thus, Theorem 3.1 corresponds to Theorem 4.1, with $k = 0$ and $\tau_1 = \tau_*$. In this section we show that the regularity result of Theorem 4.1 is in fact uniform in time, in the sense that if τ_k is as in Theorem 4.1, then $\tau_k \geqslant \tau_1$ for all $k \geqslant 1$. This implies that, if the data $\{u_0, \varphi\}$ are more regular, then the local solution $u \in X_1(\tau_*)$ found by Theorem 3.1 is in fact in higher order spaces $X_k(\tau_*)$, $k > 1$, with *the same* τ_*.

Theorem 4.2. *Let $k \geqslant 0$ and $T > 0$, and assume that problem* (P) *has a solution $u \in X_k(\tau)$, for some $\tau \in]0, T]$, corresponding to data $u_0 \in H^{m+k}$, $\varphi \in Y_k(T)$. If in addition $u_0 \in H^{m+k+1}$ and $\varphi \in Y_{k+1}(T)$, then $u \in X_{k+1}(\tau)$.*

We remark that Theorem 4.2 holds also for $k = 0$, even if we do not know whether problem (P) has a solution in $X_0(\tau)$. Also, Theorems 4.1 and 4.2, with the appropriate modifications, hold for $T = +\infty$ as well.

Proof. By Theorem 4.1, problem (P) has a local solution $u \in X_{k+1}(\tau_{k+1})$, for some $\tau_{k+1} \in]0, T]$. If $\tau_{k+1} \geqslant \tau$, there is nothing to prove. Otherwise, let T_{k+1} be the life-span of u with respect to the X_{k+1}-norm; that is,

$$T_{k+1} := \sup\big\{\theta \in]0, T] \mid u \in X_{k+1}(\theta)\big\}; \tag{4.1}$$

we need to prove that $\tau < T_{k+1}$. To this end, it is sufficient to show that the norm of u in $X_{k+1}(\tau)$ can be bounded only in terms of the norms of u in $X_k(\tau)$, $u_0 \in H^{m+k+1}$, and $\varphi \in Y_{k+1}(T)$; that is, there is a number

$$\Psi = \Psi\big(\|u\|_{X_k(\tau)}, \|u_0\|_{m+k+1}, \|\varphi\|_{Y_{k+1}(T)}\big),$$

such that

$$\|u\|_{X_{k+1}(\tau)} \leqslant \Psi. \tag{4.2}$$

Indeed, if $T_{k+1} \leqslant \tau$, (4.2) would contradict the fact that

$$\limsup_{\theta \nearrow T_{k+1}} \|u\|_{X_{k+1}(\theta)} = +\infty.$$

To show (4.2), we start by multiplying (1.4) in H^0 by $\Delta^{m+k+1}u + \Delta^{k+1}u_t$, to obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{m+k+1} + \|u\|^2_{2m+k+1} + \|u_t\|^2_{k+1} \\ &\quad = \big\langle \nabla^{k+1}\big(N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u)\big), \nabla^{k+1}(\Delta^m u + u_t)\big\rangle \\ &\quad \leqslant \big(\big\|N(f, u^{(m-1)})\big\|_{k+1} + \big\|N(\varphi^{(m-1)}, u)\big\|_{k+1}\big)\big(\|u\|_{2m+k+1} + \|u_t\|_{k+1}\big). \end{aligned} \tag{4.3}$$

To estimate $\|N(f, u^{(m-1)})\|_{k+1}$, it is sufficient to consider any m-tuple $\kappa = (k_1, \dots, k_m) \in \mathbb{N}^m$, with $k_1 + \cdots + k_m = k+1$, and estimate all products of the type

$$A_\kappa := \nabla^{k_1+2} f \cdot \nabla^{k_2+2}u \cdots \nabla^{k_m+2}u \tag{4.4}$$

in L^2. By the imbedding (1.9), with $h = 1$, and by (1.28), with $h = k_1 + 1$,

$$\begin{aligned} |A_\kappa|_2 &\leqslant |\nabla^{k_1+2} f|_{2m} |\nabla^{k_2+2}u|_{2m} \cdots |\nabla^{k_m+2}u|_{2m} \\ &\leqslant C_S \|f\|_{m+1+k_1} \|u\|_{m+1+k_2} \cdots \|u\|_{m+1+k_m} \\ &\leqslant C_S C \|u\|^{m-1}_m \prod_{j=1}^m \|u\|_{m+1+k_j}. \end{aligned} \tag{4.5}$$

If $k_j \leqslant k$ for all j, $1 \leqslant j \leqslant m$, then, by (1.11),

$$\|u\|_{m+1+k_j} \leqslant \|u\|_{m+1+k} \leqslant C_I \|u\|^{1-1/m}_{m+k} \|u\|^{1/m}_{2m+k},$$

so that, from (4.5),

$$|A_\kappa|_2 \leqslant C\|u\|^{m-1}_m \|u\|^{m-1}_{m+k} \|u\|_{2m+k} \leqslant C\|u\|^{2m-3}_{m+k} \|u\|_{2m+k} \|u\|_{m+k+1}. \tag{4.6}$$

If instead there is an index j such that $k_j = k+1$, then $k_i = 0$ for all $i \neq j$, and from (4.5),

$$|A_\kappa|_2 \leqslant C_S C \|u\|^{m-1}_m \|u\|^{m-1}_{m+1} \|u\|_{m+k+2}. \tag{4.7}$$

If $k = 0$, using interpolation, we proceed with

$$\begin{aligned} |A_\kappa|_2 &\leqslant C\|u\|^{m-1}_m \|u\|^{m-2}_{m+1} \|u\|_{m+2} \|u\|_{m+1} \\ &\leqslant C C_I \|u\|^{m-1}_m \|u\|^{\frac{(m-2)(m-1)}{m}}_m \|u\|^{\frac{m-2}{m}}_{2m} \|u\|^{\frac{m-2}{m}}_m \|u\|^{\frac{2}{m}}_{2m} \|u\|_{m+1} \\ &= C\|u\|^{2m-3}_m \|u\|_{2m} \|u\|_{m+1}, \end{aligned} \tag{4.8}$$

and note that (4.8) agrees with (4.6) when $k = 0$. If $k \geqslant 1$, we proceed from (4.7) with

$$|A_\kappa|_2 \leqslant C\|u\|^{m-1}_m \|u\|^{m-1}_{m+1} \|u\|_{2m+k} \leqslant C\|u\|^{2m-3}_{m+k} \|u\|_{m+k+1} \|u\|_{2m+k}, \tag{4.9}$$

which again agrees with (4.6). In conclusion, we deduce from (4.6), (4.8) and (4.9), that

$$\big\|N(f,u^{(m-1)})\big\|_{k+1} \leqslant C\|u\|_{m+k}^{2m-3}\|u\|_{2m+k}\|u\|_{m+k+1}. \tag{4.10}$$

We estimate $\|N(\varphi^{(m-1)},u)\|_{k+1}$ in a similar way, considering, as in (4.4), products of the type

$$B_\kappa := \nabla^{k_1+2}\varphi\cdots\nabla^{k_{m-1}+2}\varphi\cdot\nabla^{k_m+2}u \tag{4.11}$$

in L^2. As in (4.5), since each $k_j \leqslant k+1$,

$$\begin{aligned}
|B_\kappa|_2 &\leqslant C_S\|u\|_{m+1+k_m}\prod_{j=1}^{m-1}\|\varphi\|_{m+1+k_j}\\
&\leqslant C_S\|u\|_{m+k+2}\|\varphi\|_{m+k+2}^{m-1}\\
&\leqslant C_S\|u\|_{m+k+1}^{\frac{m-1}{m}}\|u\|_{2m+k+1}^{\frac{1}{m}}\|\varphi\|_{m+k+1}^{\frac{(m-1)^2}{m}}\|\varphi\|_{2m+k+1}^{\frac{m-1}{m}}\\
&\leqslant \eta\|u\|_{2m+k+1} + C_\eta\|\varphi\|_{m+k+1}^{m-1}\|\varphi\|_{2m+k+1}\|u\|_{m+k+1},
\end{aligned} \tag{4.12}$$

for $\eta > 0$. Choosing sufficiently small η, and combining (4.10) and (4.12) into (4.3), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{m+k+1}^2 + \frac{1}{2}\big(\|u\|_{2m+k+1}^2 + \|u_t\|_{k+1}^2\big)\\
&\quad\leqslant C\big(\|u\|_{m+k}^{2(2m-3)}\|u\|_{2m+k}^2 + \|\varphi\|_{m+k+1}^{2(m-1)}\|\varphi\|_{2m+k+1}^2\big)\|u\|_{m+k+1}^2;
\end{aligned}$$

thus, by Gronwall's inequality, and recalling (1.19),

$$\begin{aligned}
&\big\|u(t)\big\|_{m+k+1}^2 + \frac{1}{2}\int_0^t\big(\|u\|_{2m+k+1}^2 + \|u_t\|_{k+1}^2\big)\,\mathrm{d}t\\
&\quad\leqslant \|u_0\|_{m+k+1}^2\exp\left(C\big(\sigma_{k,\tau}(u)\big)^{2(2m-3)}\int_0^\tau\|u\|_{2m+k}^2\,\mathrm{d}t\right.\\
&\qquad\left.+\,C\big(\sigma_{k+1,T}(\varphi)\big)^{2(m-1)}\int_0^T\|\varphi\|_{2m+k+1}^2\,\mathrm{d}t\right).
\end{aligned} \tag{4.13}$$

The right side of (4.13) defines the function Ψ required in (4.1), (4.2), showing that the norm of u in $X_{k+1}(\tau)$ can be bounded only in terms of the norms of u in $X_k(\tau)$, $u_0 \in H^{m+k+1}$, and $\varphi \in Y_{k+1}(T)$, as claimed. □

Corollary 4.1. *If the data $\{u_0,\varphi\}$ are smooth, i.e. $u_0 \in C^\infty(V)$ and $\varphi \in C^\infty([0,T]\times V)$, then problem* (P) *admits a solution $u \in C^\infty([0,\tau]\times V)$, for some $\tau \in\,]0,T]$.*

Proof. From Theorem 4.2, it follows that problem (P) has a local solution $\{u,f\}$, with, at least, $u, f \in C([0,\tau];H^s)$ for all $s \geqslant 0$ (using Lemma 1.3 for f). By Eq. (1.4), it follows that $u_t \in C([0,\tau];H^s)$ as well, for all $s \geqslant 0$. We proceed then by induction: differentiating Eqs. (1.5) and (1.4) successively with respect to t, we deduce that for all integer $k \geqslant 0$,

$$\partial_t^k u \in C\big([0,\tau];H^s\big) \Longrightarrow \partial_t^k f \in C\big([0,\tau];H^s\big) \Longrightarrow \partial_t^{k+1}u \in C\big([0,\tau];H^s\big).$$

Hence, $\partial_t^k u \in C([0,\tau]; H^s)$ for all $k \geqslant 0$. Since $\bigcap_{s\geqslant 0} H^s = C^\infty(V)$ because V is compact, we can conclude that $u \in C^\infty([0,\tau]\times V)$. To see this, fix $k \geqslant 0$, a multiindex $\alpha \in \mathrm{N}^{2m}$, and a point $(t_0,x_0)\in[0,\tau]\times V$. Let $v := \partial_t^k \nabla^\alpha u$, set $a := |\alpha|$, and choose any $s > m$. Then, recalling that $H^s(V) \hookrightarrow C^{0,\lambda}(V)$ for some $\lambda \in]0,1]$:

$$\begin{aligned}
\big|v(t,x)-v(t_0,x_0)\big| &\leqslant \big|v(t,x)-v(t,x_0)\big| + \big|v(t,x_0)-v(t_0,x_0)\big| \\
&\leqslant \big|v(t,x)-v(t,x_0)\big| + \left|\int_{t_0}^{t} v_t(\theta,x_0)\,\mathrm{d}\theta\right| \\
&\leqslant \big|v(t,x)-v(t,x_0)\big| + \left|\int_{t_0}^{t} \big|v_t(\theta,\cdot)\big|_\infty\,\mathrm{d}\theta\right| \\
&\leqslant C\big\|v(t,\cdot)\big\|_s|x-x_0|^\lambda + C\left|\int_{t_0}^{t} \big\|v_t(\theta,\cdot)\big\|_s\,\mathrm{d}\theta\right| \\
&\leqslant C\big\|\partial_t^k u(t,\cdot)\big\|_{s+a}|x-x_0|^\lambda + C\left|\int_{t_0}^{t} \big\|\partial_t^{k+1} u(\theta,\cdot)\big\|_{s+a}\,\mathrm{d}\theta\right| \\
&\leqslant C\max_{0\leqslant t\leqslant\tau}\big\|\partial_t^k u(t,\cdot)\big\|_{s+a}|x-x_0|^\lambda \\
&\quad + C\max_{0\leqslant t\leqslant\tau}\big\|\partial_t^{k+1} u(t,\cdot)\big\|_{s+a}|t-t_0| \\
&\leqslant C_{s,k}\big(|x-x_0|^\lambda + |t-t_0|\big).
\end{aligned}$$

This shows the continuity of $\partial_t^k\nabla^\alpha u$ at (t_0,x_0); since (t_0,x_0), k and α are arbitrary, we conclude that $u \in C^\infty([0,\tau]\times V)$, as asserted. □

5. Global existence

In this section we prove a global existence result for solutions of problem (P) in $X_0(+\infty)$, corresponding to data $(u_0,\varphi)\in H^m\times Y_0(+\infty)$ that are sufficiently small. This result is proven by first showing that problem (P) has a solution $u\in X_0(T)$, for arbitrary $T>0$, and then by showing that u satisfies an estimate independent of T. By a standard continuation argument, this estimate allows us to extend u to a global, bounded solution of problem (P). Together with the regularity Theorem 4.2, the time-independent estimate also implies that, if $k\geqslant 1$ and the data (u_0,φ) are in $H^{m+k}\times Y_k(+\infty)$ and have sufficiently small norm in $H^m\times Y_0(+\infty)$, local solutions $u\in X_0(\tau)$ of problem (P) can be extended to global, bounded ones in $X_k(+\infty)$. In fact, arguing as in Corollary 4.1, if $u_0\in C^\infty(V)$, $\varphi\in C^\infty([0,+\infty[\times V)$, and $\|u_0\|_m+\|\varphi\|_{Y_0(+\infty)}$ is sufficiently small, then problem (P) admits a global solution $u\in C^\infty([0,+\infty[\times V)$.

Given $\varphi\in Y_0(+\infty)$ and $u_0\in H^m$, we set

$$Q_0(\varphi) := \big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\int_0^{+\infty}\|\varphi\|_{2m}^2\,\mathrm{d}t, \tag{5.1}$$

and, for $C_0 > 0$ a constant depending on V, but not on t, u_0 and φ,

$$\Lambda(u_0, \varphi) := 2\|u_0\|_m^2 \exp\big(C_0 Q_0(\varphi)\big). \tag{5.2}$$

Theorem 5.1. *Let $u_0 \in H^m$ and $\varphi \in Y_0(+\infty)$, and define $\Lambda(u_0, \varphi)$ by* (5.2). *There exist Λ_0 and $R_0 > 0$ such that, if $\Lambda(u_0, \varphi) \leqslant \Lambda_0$, problem* (P) *has a unique solution $u \in X_0(+\infty)$, which satisfies the bound*

$$\|u\|^2_{X_0(+\infty)} = \int_0^{+\infty} \big(\|u\|_{2m}^2 + \|u_t\|_0^2\big)\,\mathrm{d}t \leqslant R_0^2. \tag{5.3}$$

In addition, there is $\Lambda_1 \leqslant \Lambda_0$ such that, if $\Lambda(u_0, \varphi) \leqslant \Lambda_1$,

$$2\big\|u(t)\big\|_m^2 \leqslant \Lambda(u_0, \varphi)\mathrm{e}^{-t/4}. \tag{5.4}$$

Proof. Uniqueness of solutions in $X_0(T)$, for arbitrary $T > 0$, was proven in Theorem 2.1. To prove existence, we proceed in three steps, as follows. We consider again the linearized problem (LP), with

$$w \in B_0(T, R) := \big\{u \in X_0(T) \mid \big(\sigma_{0,T}(u)\big)^2 + \|u\|^2_{X_0(T)} \leqslant R^2\big\}, \tag{5.5}$$

where R is now to be determined, but $T > 0$ is arbitrary. Referring to the map $\Phi : w \mapsto u$ defined by (3.8), our first step is to prove

Theorem 5.2. *There exists $\Lambda_0 > 0$ such that if $\Lambda(u_0, \varphi) \leqslant \Lambda_0$, there is $R_0 > 0$ such that, for all $T > 0$, Φ maps the ball $B_0(T, R_0)$ into itself.*

Our second step is to deduce from Theorem 5.2 that problem (P) has a solution $u \in X_0(T)$, satisfying the bound

$$\big(\sigma_{0,T}(u)\big)^2 + \|u\|^2_{X_0(T)} \leqslant R_0^2. \tag{5.6}$$

Since R_0 is independent of T, (5.6) implies that u can be extended to all of $[0, +\infty[$, into a solution $u \in X_0(+\infty)$ which satisfies the global bound (5.3). Our last step is then to prove the decay estimate (5.4).

Step 1: Proof of Theorem 5.2. Multiplying Eq. (3.5) in H^0 by $\Delta^m u + u_t$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 + \|u\|_{2m}^2 + \|u_t\|_0^2 = \big\langle N(f, w^{(m-2)}, u) + N(\varphi^{(m-1)}, u), \Delta^m u + u_t\big\rangle. \tag{5.7}$$

As in Section 3.2, recalling (1.22) and Lemma 1.3, we start to estimate

$$\begin{aligned}
A &:= \big\langle N(f, w^{(m-2)}, u), \Delta^m u + u_t\big\rangle \\
&\leqslant C_H C_S \|f\|_{m+1} \|w\|_{m+1}^{m-2} \|u\|_{m+1} \big(\|u\|_{2m} + \|u_t\|_0\big) \\
&\leqslant C_H C_S \|w\|_m^{m-1} \|w\|_{m+1}^{m-1} \|u\|_{m+1} \big(\|u\|_{2m} + \|u_t\|_0\big) \\
&\leqslant C_H C_S C_I \|w\|_m^a \|w\|_{2m}^{(m-1)/m} \|u\|_m^{(m-1)/m} \|u\|_{2m}^{1/m} \big(\|u\|_{2m} + \|u_t\|_0\big),
\end{aligned} \tag{5.8}$$

with $a = (m-1)(2m-1)/m$. Since $w \in B_0(T, R)$, by Hölder's inequality we deduce from (5.8)

$$|A| \leqslant C R^a \|w\|_{2m}^{(m-1)/m} \|u\|_m^{(m-1)/m} \|u\|_{2m}^{1/m} \big(\|u\|_{2m} + \|u_t\|_0\big)$$
$$\leqslant \eta\big(\|u\|_{2m}^2 + \|u_t\|_0^2\big) + C_\eta R^{2(2m-1)} \|w\|_{2m}^2 \|u\|_m^2. \tag{5.9}$$

Similarly,

$$B := \big\langle N(\varphi^{(m-1)}, u), \Delta^m u + u_t\big\rangle$$
$$\leqslant C C_S \|\varphi\|_{m+1}^{m-1} \|u\|_{m+1} \big(\|u\|_{2m} + \|u_t\|_0\big)$$
$$\leqslant C C_S C_I \|\varphi\|_m^{\frac{(m-1)^2}{m}} \|\varphi\|_{2m}^{\frac{m-1}{m}} \|u\|_m^{\frac{m-1}{m}} \|u\|_{2m}^{1/m} \big(\|u\|_{2m} + \|u_t\|_0\big)$$
$$\leqslant \eta\big(\|u\|_{2m}^2 + \|u_t\|_0^2\big) + C_\eta \big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)} \|\varphi\|_{2m}^2 \|u\|_m^2. \tag{5.10}$$

Choosing η small, and setting $C_\eta =: C_0$, we deduce from (5.7)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 + \frac{1}{2}\big(\|u\|_{2m}^2 + \|u_t\|_0^2\big)$$
$$\leqslant C_0\big(R^{2(2m-1)}\|w\|_{2m}^2 + \big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\|\varphi\|_{2m}^2\big)\|u\|_m^2. \tag{5.11}$$

By Gronwall's inequality, for $0 \leqslant t \leqslant T$, with T arbitrary,

$$\big\|u(t)\big\|_m^2 + \frac{1}{2}\int_0^t \big(\|u\|_{2m}^2 + \|u_t\|_0^2\big)\,\mathrm{d}\theta$$
$$\leqslant \|u_0\|_m^2 \exp\left(C_0\left(R^{2(2m-1)}\int_0^T \|w\|_{2m}^2\,\mathrm{d}\theta + \big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\int_0^\infty \|\varphi\|_{2m}^2\,\mathrm{d}\theta\right)\right)$$
$$\leqslant \|u_0\|_m^2 \exp\big(C_0 Q_0(\varphi)\big)\exp(C_0 R^{4m}),$$

having kept in mind that $w \in B_0(T, R)$, and the definition (5.1) of $Q_0(\varphi)$. Consequently, recalling (5.2),

$$\big\|u(t)\big\|_m^2 + \int_0^t \big(\|u\|_{2m}^2 + \|u_t\|_0^2\big)\,\mathrm{d}\theta \leqslant \Lambda(u_0, \varphi)\mathrm{e}^{C_0 R^{4m}}. \tag{5.12}$$

Thus, to prove Theorem 5.2, it is sufficient to find $R_0 > 0$ such that

$$\Lambda(u_0, \varphi)\mathrm{e}^{C_0 R_0^{4m}} \leqslant R_0^2. \tag{5.13}$$

Abbreviating $\Lambda(u_0, \varphi) =: \Lambda$, and setting $R^2 =: r$, we need to solve the inequality

$$h(r) := r - \Lambda \mathrm{e}^{C_0 r^{2m}} \geqslant 0. \tag{5.14}$$

Since

$$h'(r) = 1 - 2mC_0\Lambda r^{2m-1}\mathrm{e}^{C_0 r^{2m}},$$

h has a maximum at $r_\mu > 0$, solution of the equation

$$2mC_0\Lambda r^{2m-1}\mathrm{e}^{C_0 r^{2m}} = 1. \tag{5.15}$$

We wish to show that $h(r_\mu) > 0$ if Λ is sufficiently small. From (5.14) and (5.15),

$$h(r_\mu) = r_\mu - \frac{1}{2mC_0 r_\mu^{2m-1}}; \tag{5.16}$$

thus, if $h(r_\mu) \leqslant 0$ for all Λ, by (5.16) $2mC_0 r_\mu^{2m} \leqslant 1$, so from (5.15)

$$1 = 2mC_0 \Lambda r_\mu^{2m-1} \mathrm{e}^{C_0 r_\mu^{2m}} \leqslant (2mC_0\mathrm{e})^{1/2m} \Lambda.$$

This leads to a contradiction as $\Lambda \to 0$; consequently, there is $\Lambda_0 > 0$ such that the corresponding r_μ, defined by (5.15) with $\Lambda = \Lambda_0$, satisfies $h(r_\mu) > 0$. Hence, (5.13) holds for all $\Lambda \leqslant \Lambda_0$, if $R_0 := \sqrt{r_\mu}$. This concludes the proof of Theorem 5.2. □

Step 2: Proof of Theorem 5.1, I. We follow an argument similar to the proof of Theorem 3.1. Indeed, since the sequence $(u^n)_{n\geqslant 0}$ defined by (3.9) is contained in $B_0(T, R_0)$, there is $u \in B_0(T, R_0)$ such that, just as in (3.10), (3.11), (3.13) and (3.14),

$$u^n \to u \quad \text{weakly in } L^2(0,T;H^{2m}), \tag{5.17}$$

$$u_t^n \to u_t \quad \text{weakly in } L^2(0,T;H^{0}), \tag{5.18}$$

$$u^n \to u \quad \text{strongly in } L^2(0,T;H^{2m-1}), \tag{5.19}$$

$$u^n \to u \quad \text{strongly in } C([0,T];H^{m-1}). \tag{5.20}$$

We now show that (3.15) can be replaced by

$$M(u^n) \to M(u) \quad \text{in } L^2(0,T;L^p),\ \ p := \frac{2m}{2m-1}. \tag{5.21}$$

Indeed, recalling (3.16), since $\frac{k-1}{m} + \frac{1}{2m} + \frac{m-k}{m} = \frac{1}{p}$ we can modify estimate (3.17) into

$$\begin{aligned} |D_k|_p &\leqslant C_H |\nabla^2 u|_m^{k-1} |\nabla^2 z^n|_{2m} |\nabla^2 u^n|_m^{m-k} \\ &\leqslant C_H C_S \|u\|_m^{k-1} \|z^n\|_{m+1} \|u^n\|_m^{m-k} \\ &\leqslant C_H C_S R_0^{m-1} \|z^n\|_{m+1}; \end{aligned} \tag{5.22}$$

thus, (5.21) follows from (5.19), which implies that $u^n \to u$ also in $L^2(0,T;H^{m+1})$. Defining again $f \in Y_0(T)$ by (1.5), we can then deduce from (3.18) and (5.21) that

$$f^n \to f \quad \text{in } L^2(0,T;H^{m+1}); \tag{5.23}$$

in fact, we compute that

$$\begin{aligned} \|f^n - f\|_{m+1}^2 &= \langle \Delta^m (f^n - f), \Delta(f^n - f)\rangle = \langle M(u) - M(u^n), \Delta(f^n - f)\rangle \\ &\leqslant \big|M(u) - M(u^n)\big|_p \big|\Delta(f^n - f)\big|_{2m} \leqslant \big|M(u) - M(u^n)\big|_p \|f^n - f\|_{m+1}; \end{aligned}$$

thus,

$$\|f^n - f\|_{m+1} \leqslant C\big|M(u) - M(u^n)\big|_p,$$

and (5.23) follows from (5.21). Finally, we show that

$$N\big(f^n, (u^n)^{(m-2)}, u^{n+1}\big) \to N(f, u^{(m-1)}) \quad \text{in } L^2(0,T;L^p). \tag{5.24}$$

Indeed, defining $N_n^{(j)}$ as in (3.21), $j = 1, 2, 3$, we can replace (3.22) by

$$\begin{aligned} |N_n^{(1)}|_p &\leqslant C_H C_S \|f^n - f\|_{m+1} \|u^n\|_m^{m-2} \|u^{n+1}\|_m \\ &\leqslant C_H C_S R_0^{m-1} \|f^n - f\|_{m+1}; \end{aligned} \tag{5.25}$$

analogously, and recalling that, by (1.28) of Lemma 1.3,

$$\|f^n(t)\|_m \leqslant C\|u^n(t)\|_m^m \leqslant CR_0^m \tag{5.26}$$

for all $t \in [0, T]$, we can also replace (3.23) and (3.24) by

$$|N_n^{(2)}|_p \leqslant CC_H C_S R_0^{2m-2} \|z^n\|_{m+1}, \tag{5.27}$$

$$|N_n^{(3)}|_p \leqslant CC_H C_S R_0^{2m-2} \|z^{n+1}\|_{m+1}. \tag{5.28}$$

Hence, (5.24) follows, by (5.19). Since $1 < p < 2$, (5.17) implies that

$$\Delta^m u^n \to \Delta^m u \quad \text{weakly in } L^2(0, T; L^p); \tag{5.29}$$

thus, as in (3.25) and by (5.24),

$$u_t^n \to -\Delta^m u + N(f, u^{(m-1)}) + N(\varphi^{(m-1)}, u) \quad \text{weakly in } L^2(0, T; L^p) \tag{5.30}$$

as well. Comparing this to (5.18), we conclude that u satisfies Eq. (1.4); moreover, u satisfies the initial condition (1.6), because, by (5.20), $u^n(0) = u_0 \to u(0)$ in H^{m-1}. Finally, (5.6) follows from the fact that $u \in B_0(T; R_0)$. This completes the proof of Theorem 5.1, up to (5.3). □

Step 3: Proof of Theorem 5.1, II. To prove the decay estimate (5.4), we consider the function h defined in (5.14). Since $h(0) = -\Lambda < 0$, and h is strictly increasing on $[0, r_\mu]$, there is a unique $r_0 \in]0, r_\mu]$ such that $h(r_0) = 0$. We claim that $r_0 \to 0$ as $\Lambda \to 0$. Indeed, if there were $\varepsilon_0 > 0$ such that $r_0 \geqslant \varepsilon_0$ for some sequence $(\Lambda_n)_{n\geqslant 1}$, with $\Lambda_n \to 0$, then from (5.14) and (5.15),

$$\varepsilon_0 \leqslant r_0 = \Lambda_n \mathrm{e}^{C_0 r_0^{2m}} \leqslant \Lambda_n \mathrm{e}^{C_0 r_\mu^{2m}} = (2mC_0 r_\mu^{2m-1})^{-1}. \tag{5.31}$$

However, (5.15) implies that $r_\mu \to +\infty$ as $\Lambda \to 0$, which is in contradiction with (5.31). Thus, there is $\Lambda_1 \in]0, \Lambda_0]$ such that the corresponding r_0 verifies

$$4C_0 r_0^{2m} \leqslant 1. \tag{5.32}$$

Choosing then $R := \sqrt{r_0}$ in (5.5), we deduce from (5.12), and the definition of r_0, that if $\Lambda \leqslant \Lambda_1$, then for all $t \geqslant 0$,

$$\|u(t)\|_m^2 \leqslant \Lambda \mathrm{e}^{C_0 R^{4m}} = R^2 \leqslant (4C_0)^{-1/2m}. \tag{5.33}$$

Next, we note that u satisfies estimate (5.11), with $w = u$; thus, by (5.33) and (5.32),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 + \frac{1}{2}\big(\|u\|_{2m}^2 + \|u_t\|_0^2\big) \\
&\quad \leqslant C_0 R^{4m}\|u\|_{2m}^2 + C_0\big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\|\varphi\|_{2m}^2\|u\|_m^2 \\
&\quad \leqslant \frac{1}{4}\|u\|_{2m}^2 + C_0\big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\|\varphi\|_{2m}^2\|u\|_m^2,
\end{aligned} \tag{5.34}$$

from which, since $\|u\|_m \leqslant \|u\|_{2m}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_m^2 + \frac{1}{4}\|u\|_m^2 \leqslant C_0\big(\sigma_{0,\infty}(\varphi)\big)^{2(m-1)}\|\varphi\|_{2m}^2\|u\|_m^2.$$

This implies

$$\mathrm{e}^{t/4}\|u(t)\|_m^2 \leqslant \|u_0\|_m^2 \mathrm{e}^{C_0 Q_0(\varphi)},$$

and (5.4) follows. The proof of Theorem 5.1 is now complete. □

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